Assignment 2.

This homework is due *Tuesday*, September 10.

There are total 22 points in this assignment. 17 points is considered 100%. If you go over 17 points, you will get over 100% for this homework and it will count towards your course grade (but not over 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 1.2-1.3 and a part of 2.1 in Bartle–Sherbert.

1. Exercises

(1) [2pt] (Paragraph 1.2.4g) Find a mistake in the following (erroneous!) arguments:
Claim: If $n \in \mathbb{N}$ and if the maximum of the natural numbers p, q is n, then $p = q$.
"Proof." Proof by induction in n. Evidently, for $n = 1$ claim is true since in such case, $p = 1$ and $q = 1$.
Suppose, the claim holds for some $n \in \mathbb{N}$. Prove that then it also holds for $n+1$. Suppose maximum of p and q is $n+1$. Then maximum of $p-1$ and $q-1$ is $(n+1)-1 = n$. By induction hypothesis, $p-1 = q-1$, therefore $p = q$. Thus, the claim holds for $n+1$ and, by induction principle, for all natural numbers.

(2) [2pt] (1.3.4) Exhibit (define explicitly) a bijection from N to the set of all odd integers greater than 13.

(3) Exhibit (define explicitly) a bijection between
   (a) [2pt] $\mathbb{Z}$ and $\mathbb{Z} \setminus \{0\}$,
   (b) [3pt, optional[1]] $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$,

(4) [3pt] (1.3.13) Prove that the collection $\mathcal{F}(\mathbb{N})$ of all *finite* subsets of $\mathbb{N}$ is countable.

— *see next page* —

[1]That is, not included in denominator of the grade.

2. Quick cheat-sheet

Reminder. (Subsection 2.1.1) On the set $\mathbb{R}$ of real numbers there two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) $a + b = b + a$ for all $a, b \in \mathbb{R}$,
(A2) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$,
(A3) there exists $0 \in \mathbb{R}$ s.t. $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$,
(A4) for each $a \in \mathbb{R}$ there exists an element $-a$ s.t. $a+(-a) = (-a)+a = 0$,
(M1) $ab = ba$ for all $a, b \in \mathbb{R}$,
(M2) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$,
(M3) there exists $1 \in \mathbb{R}$ s.t. $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$,
(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ s.t. $a\cdot(1/a) = (1/a)\cdot a = 1$,
(D) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in \mathbb{R}$.

3. Exercises

(5) (a) [1pt] Is addition of real numbers distributive over multiplication?

(b) [1pt] Is set union distributive over set intersection? That is, is it true that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets A, B, C?

(c) [1pt] Is set intersection distributive over set union?

(6) On the set $\mathbb{N}$, consider two operations: $\oplus$ and $\odot$ defined as follows: $a \oplus b = ab$ and $a \odot b = a^b$.

(a) [1pt] Do properties A1, A2 hold for $\oplus$? That is, is it true that $a \oplus b = b \oplus a$, and that $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all $a, b, c \in \mathbb{N}$?
(*Hint:* For this and further items, the main way to figure out questions is to write out expressions with $\odot$ and $\oplus$ in terms of "usual" operations, using definition of $\odot$ and $\oplus$.)

(b) [1pt] Do properties M1, M2 hold for $\odot$?

(c) [1pt] Is there unit element with respect to $\odot$? That is, is there an element $1_\odot \in \mathbb{N}$ such that $1_\odot \odot a = a \odot 1_\odot = a$ for all $a \in \mathbb{N}$?

(d) [1pt] Is there a *right* unit element with respect to $\odot$? That is, is there an element $1_r \in \mathbb{N}$ such that $a \odot 1_r = a$ for all $a \in \mathbb{N}$?

(e) [1pt] Is there a *left* unit element with respect to $\odot$? That is, is there an element $1_\ell \in \mathbb{N}$ such that $1_\ell \odot a = a$ for all $a \in \mathbb{N}$?

(f) [1pt] Is $\odot$ distributive over $\oplus$ *on the left*? That is, is it true that $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$?

(g) [1pt] Is $\odot$ distributive over $\oplus$ *on the right*? That is, is it true that $c \odot (a \oplus b) = (c \odot a) \oplus (c \odot b)$?